

# QUASI-DUO DIFFERENTIAL POLYNOMIAL RINGS

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**ABSTRACT.** In this article we give a characterization of left (right) quasi-duo differential polynomial rings. In particular, we show that a differential polynomial ring is left quasi-duo if and only if it is right quasi-duo. This yields a partial answer to a question posed by Lam and Dugas in 2005. We provide non-trivial examples of such rings and give a complete description of the maximal ideals of an arbitrary quasi-duo differential polynomial ring. Moreover, we show that there is no left (right) quasi-duo differential polynomial ring in several indeterminates.

## 1. INTRODUCTION

Throughout this article, all rings are assumed to be unital and associative. Following [3], a ring  $S$  is said to be *left (right) duo* if every left (right) ideal of  $S$  is a two-sided ideal. More generally,  $S$  is said to be *left (right) quasi-duo* if every maximal left (right) ideal of  $S$  is a two-sided ideal (see e.g. [13]) or, equivalently, if every left (right) primitive homomorphic image of  $S$  is a division ring (see e.g. [11, Proposition 4]). A ring which is both left and right quasi-duo is called *quasi-duo*.

Quasi-duo rings appear in various places in ring theory, e.g. in the investigation of the Köthe conjecture (see e.g. [10, Proposition 2.5]). There are many open problems concerning left (right) quasi-duo rings, one of which is due to Lam and Dugas who ask whether there exists a right quasi-duo ring which is not left quasi-duo (see [6, Question 7.7]). Our main result (Theorem 1.1) shows that such an example can not be found in the class of differential polynomial rings.

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Recall that an *Ore extension*  $R[x; \sigma, \delta]$  is constructed from a ring  $R$ , a ring endomorphism  $\sigma : R \rightarrow R$  (respecting  $1_R$ ) and a  $\sigma$ -derivation  $\delta : R \rightarrow R$ , i.e. an additive map satisfying

$$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s, \quad \forall r, s \in R.$$

As a left  $R$ -module  $R[x; \sigma, \delta]$  is equal to the usual polynomial ring  $R[x]$ . The multiplication on  $R[x; \sigma, \delta]$  is defined by the rule

$$xr = \sigma(r)x + \delta(r)$$

for  $r \in R$ . This turns the Ore extension  $R[x; \sigma, \delta]$  into a unital and associative ring (see e.g. [9]). If  $\delta = 0$ , then  $R[x; \sigma, 0]$  is said to be a *skew polynomial ring*. If, on the other hand,  $\sigma = \text{id}_R$ , then  $\delta$  is called a *derivation* and  $R[x; \text{id}_R, \delta]$  is said to be a *differential polynomial ring* and will simply be denoted by  $R[x; \delta]$ .

In [7], Leroy, Matczuk and Puczyłowski obtained a complete characterization of left (right) quasi-duo skew polynomial rings. In [8], the same authors continued their investigation and gave a complete characterization of left (right) quasi-duo  $\mathbb{Z}$ -graded rings.

In this article we direct our attention to another type of Ore extensions, namely the differential polynomial rings. Our main result is the following.

**Theorem 1.1.** *Let  $S = R[x; \delta]$  be a differential polynomial ring, and put  $J_0 = J(S) \cap R$ . The following five assertions are equivalent:*

- (i)  *$S$  is left quasi-duo;*
- (ii)  *$S$  is right quasi-duo;*
- (iii) *Every left ideal of  $S$  containing the Jacobson radical  $J(S)$  is two-sided, i.e.  $S/J(S)$  is left duo;*
- (iv) *Every right ideal of  $S$  containing the Jacobson radical  $J(S)$  is two-sided, i.e.  $S/J(S)$  is right duo;*
- (v) *The quotient ring  $R/J_0$  is commutative and  $\delta(R) \subseteq J_0$ .*

This result provides a complete characterization of left (right) quasi-duo differential polynomial rings. In particular, it shows that a differential polynomial ring is left quasi-duo if and only if it is right quasi-duo, thereby yielding a partial answer to [6, Question 7.7]. Notice that for skew polynomial rings, which were studied in [7], the same question is still wide open.

This article is organized as follows.

In Section 2 we prove Theorem 1.1. In Section 3 we show that if  $R[x; \delta]$  is quasi-duo, then if  $R$  belongs to certain classes of rings, we can conclude that  $R[x; \delta]$  is commutative (see Proposition 3.2). This means, in particular, that  $\delta = 0$  and hence  $R[x; \delta]$  is a polynomial ring. We also provide examples of quasi-duo differential polynomial rings which are not polynomial rings (see Example 3.3). In Section 4 we give a complete description of the maximal ideals of quasi-duo differential polynomial rings (see Theorem 4.3). In Section 5 we consider differential polynomial rings in several indeterminates,  $R[X; D]$ , defined by a (countable) set of variables  $X$  and a family  $D$  of derivations on  $R$ . We show that  $R[X; D]$  can never be quasi-duo if  $X$  consist of more than one variable (see Theorem 5.3).

## 2. PROOF OF THE MAIN RESULT

In this section we give a proof of our main result, Theorem 1.1. We begin by showing that a left (right) quasi-duo differential polynomial ring over a simple ring is necessarily a polynomial ring.

**Lemma 2.1.** *Let  $S = R[x; \delta]$  be a left (right) quasi-duo differential polynomial ring. If  $R$  is a simple ring, then  $R$  is a field and  $\delta = 0$ .*

*Proof.* Let  $L$  be a maximal left (right) ideal of  $S$  containing  $x$ . By assumption,  $L$  is a two-sided ideal of  $S$  and hence, for any  $r \in R$ , we get  $\delta(r) = xr - rx \in L \cap R$ . Notice that  $L \cap R$  is a two-sided ideal of  $R$ . Using that  $R$  is a simple ring and that  $L \neq S$ , we conclude that  $L \cap R = \{0\}$  and hence  $\delta(r) = 0$ , for all  $r \in R$ .

Take  $a \in R$  and let  $M$  be a maximal left (right) ideal of  $S$  containing  $x - a$ . By assumption  $M$  is a two-sided ideal of  $S$  and hence, for any  $b \in R$ , we get  $ab - ba = b(x - a) - (x - a)b \in M$ . Using the same argument as before we get  $M \cap R = \{0\}$  and hence  $ab - ba = 0$ . This shows that  $R$  is a commutative and simple ring, i.e. a field.  $\square$

The following proposition gives us a necessary condition on the ring  $R$  and the derivation  $\delta$  in order for the differential polynomial ring  $R[x; \delta]$  to be left (right) quasi-duo.

**Proposition 2.2.** *Let  $S = R[x; \delta]$  be a differential polynomial ring, and put  $J_0 = J(S) \cap R$ . If  $S$  is left (right) quasi-duo, then  $R/J_0$  is commutative and  $\delta(R) \subseteq J_0$ .*

*Proof.* Suppose that  $S$  is left quasi-duo. (The right quasi-duo case is treated analogously.) There are now two cases: (1) there exists a maximal left ideal  $M$  of  $S$  such that  $M \cap R = \{0\}$ ; or (2)  $M \cap R \neq \{0\}$  for any non-zero maximal left ideal  $M$  of  $S$ .

**Case 1.** Suppose that there exists a maximal left ideal  $M$  of  $S$  such that  $M \cap R = \{0\}$ . In this case,  $J_0 = J(S) \cap R = \{0\}$ . We claim that  $R$  is a division ring. Indeed, since  $S$  is left quasi-duo,  $M$  is two-sided. Hence, the factor ring  $K = S/M$  is a division ring. Let  $a \in R \setminus \{0\}$ . Since  $a \notin M$ ,  $\bar{a}$  is invertible in  $K$ . Let  $a_0, a_1, \dots, a_n \in R$  such that  $\bar{a}(\bar{a}_0 + \bar{a}_1x + \dots + \bar{a}_nx^n) = \bar{1}$  in  $K$ . Then  $\bar{a}\bar{a}_0 = \bar{1}$  in  $K$ . Hence,  $aa_0 - 1 \in M$ . By the assumption  $R \cap M = \{0\}$ , we get that  $aa_0 = 1$ . This means that every non-zero element of  $R$  has a right inverse in  $R$ , which implies that  $R$  is a division ring. By Lemma 2.1 we now conclude that  $R$  is a field and that  $\delta = 0$ .

**Case 2.** Suppose that for any non-zero maximal left ideal  $M$  of  $S$ ,  $M \cap R \neq \{0\}$  holds. Let  $M$  be a non-zero maximal left ideal of  $S$ . Put  $M_0 = M \cap R$ . For any  $a \in M_0$ , one has  $\delta(a) = xa - ax \in M$ . Hence,  $\delta(a) \in M_0$  for any  $a \in M_0$ . This means that  $M_0$  is  $\delta$ -invariant. Therefore,  $\delta$  induces a derivation  $\bar{\delta}$  on  $R/M_0$ , namely,  $\bar{\delta}(\bar{a}) = \overline{\delta(a)}$ , for  $a \in R$ . Consider the map

$$\varphi : R[x; \delta] \rightarrow (R/M_0)[x; \bar{\delta}], \quad a_0 + a_1x + \dots + a_nx^n \mapsto \bar{a}_0 + \bar{a}_1x + \dots + \bar{a}_nx^n.$$

It is easy to check that  $\varphi$  is a surjective ring morphism. Thus, in view of [6, Page 245],  $(R/M_0)[x; \bar{\delta}]$  is left quasi-duo. Moreover, the ideal  $\bar{M} = \varphi(M)$  is a maximal left ideal of  $(R/M_0)[x; \bar{\delta}]$  and  $(R/M_0) \cap \bar{M} = \{\bar{0}\}$ . Now, by applying Case 1, we get that  $R/M_0$  is commutative and that  $\bar{\delta} = 0$ .

It remains to show that  $R/J_0$  is commutative and that  $\delta(R) \subseteq J_0$  holds. To see this, notice that we have already proved that for any maximal left ideal  $M$ ,  $\bar{\delta}(\bar{a}) = \bar{0}$  and  $\overline{ab} = \overline{ba}$  for any  $\bar{a}, \bar{b} \in R/M_0$  where  $M_0 = M \cap R$ . As a corollary,  $\delta(a) \in M$  and  $ab - ba \in M$ , for all  $a, b \in R$  and every maximal left ideal  $M$  of  $S$ . Therefore,  $\delta(a), ab - ba \in J(S) \cap R = J_0$  for any  $a, b \in R$ . Thus,  $\delta(R) \subseteq J_0$  and  $R/J_0$  is commutative. This concludes the proof.  $\square$

We shall now prove Theorem 1.1 and thereby get a complete characterization of quasi-duoness of  $R[x; \delta]$ .

### Proof of Theorem 1.1

We will only prove the left sided case, i.e. (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v). The right sided case, i.e. (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v), is treated analogously. We will now show that (i) $\Rightarrow$ (v) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (v): This implication follows from Proposition 2.2.

(v) $\Rightarrow$ (iii): Consider the morphism  $\varphi$  as defined in Case 2 of Proposition 2.2:

$$\varphi : R[x; \delta] \rightarrow (R/J_0)[x; \bar{\delta}], \quad a_0 + a_1x + \dots + a_nx^n \mapsto \bar{a}_0 + \bar{a}_1x + \dots + \bar{a}_nx^n.$$

In our case,  $R/J_0$  is commutative and  $\bar{\delta} = 0$ . Hence,  $(R/J_0)[x, \bar{\delta}]$  is commutative. Notice that  $\varphi$  is surjective and that  $\ker(\varphi) = J_0[x, \delta] \subseteq J(S)$ . Hence,  $S/(J_0[x, \delta]) \cong (R/J_0)[x, \bar{\delta}]$  which is commutative. Therefore, every left ideal of  $S$  containing  $J(S)$  is two-sided.

(iii) $\Rightarrow$ (i): This is trivial.  $\square$

**Remark 2.3.** Theorem 1.1 shows that a differential polynomial ring is left quasi-duo if and only if it is right quasi-duo. Henceforth, we need not make a distinction between the left and the right properties and shall simply use the notion *quasi-duo*.

### 3. TRIVIAL AND NON-TRIVIAL QUASI-DUO DIFFERENTIAL POLYNOMIAL RINGS

By Lemma 2.1 we have observed that if the differential polynomial ring  $R[x; \delta]$  is quasi-duo and  $R$  is simple, then  $R[x; \delta]$  is necessarily a commutative polynomial ring. In this section we show that the same conclusion holds for large classes of rings  $R$  which are not necessarily simple (see Proposition 3.2). We will also show that there exist quasi-duo differential polynomial rings which are not classical polynomial rings (see Example 3.3).

**Lemma 3.1.** *Let  $S = R[x; \delta]$  be a differential polynomial ring, and denote the nilradical of  $R$  by  $\text{Nil}(R)$ . Suppose that  $R$  satisfies at least one of the following conditions:*

- (i)  $\text{Nil}(R) = \{0\}$  and  $R$  is a PI-ring, i.e.  $R$  satisfies a polynomial identity;
- (ii)  $\text{Nil}(R) = \{0\}$  and  $R$  satisfies the ascending chain condition on right annihilators.

*Then,  $S$  is semiprimitive, i.e.  $J(S) = \{0\}$ .*

*Proof.* This is just a corollary of [12].  $\square$

By the preceding lemma, the class of rings  $R$  over which differential polynomial rings  $R[x; \delta]$  are semiprimitive includes e.g. semiprime commutative rings, domains, and noetherian rings with  $\text{Nil}(R) = \{0\}$ .

**Proposition 3.2.** *Let  $R$  be a ring satisfying  $\text{Nil}(R) = \{0\}$ . If  $R$  is a PI-ring or satisfies the ascending chain condition on right annihilators, then the following two assertions are equivalent:*

- (i)  $R[x; \delta]$  is quasi-duo;
- (ii)  $R[x; \delta]$  is commutative.

*Proof.* The desired conclusion follows from Lemma 3.1 and Theorem 1.1.  $\square$

The following example demonstrates a quasi-duo differential polynomial ring which is non-trivial, i.e. not a polynomial ring.

**Example 3.3.** Let  $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$ . It is not difficult to see that the Jacobson radical of  $R$  is  $J(R) = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \mid b \in \mathbb{R} \right\}$ . Hence,  $R/J(R) = \left\{ \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \mid a, c \in \mathbb{R} \right\}$  is commutative. Take  $A \in J(R) \setminus \{0\}$  and let  $\delta$  be the inner derivation on  $R$  defined by  $\delta(B) = AB - BA$ , for  $B \in R$ . Clearly,  $\delta(R) \subseteq J(R)$ . Now consider the corresponding differential polynomial ring  $R[x; \delta]$ . Using that  $R$  is a PI-ring over  $\mathbb{R}$ , a field of characteristic zero, [1, Theorem 1.2] yields that  $J(R[x; \delta]) = \text{Nil}(R)[x; \delta]$ . Since every element of  $R$  is a root of a non-zero polynomial over  $\mathbb{R}$ , by [5, Corollary 4.19],  $J(R) = \text{Nil}(R)$ . This means that  $J_0 = J(R[x; \delta]) \cap R = J(R)$ . Therefore,  $R[x; \delta]$  satisfies Theorem 1.1(v) and hence  $R[x; \delta]$  is left and right quasi-duo.

**Remark 3.4.** One may replace  $R$  by an arbitrary ring of upper triangular  $n$  by  $n$  matrices and mimic the above construction. This gives us a whole family of non-trivial quasi-duo differential polynomial ring.

#### 4. MAXIMAL IDEALS AND THE JACOBSON RADICAL

In this section, we shall describe all maximal ideals of  $S = R[x; \delta]$  in the case when  $S$  is quasi-duo. Notice that in [7], Leroy et al. gave a complete characterization of left (right) quasi-duo skew polynomial rings  $R[x; \sigma, 0]$  where  $\sigma$  is an automorphism of  $R$ . In order to do so, they first described all maximal ideals of  $R[x; \sigma, 0]$ , and then used their description to characterize quasi-duoness. In this article we work in the opposite direction. In fact, we use our main result (Theorem 1.1) to find all maximal ideals of  $S = R[x; \delta]$  as well as the Jacobson radical of  $R[x; \delta]$ .

**Remark 4.1.** Suppose that  $R[x; \delta]$  is quasi-duo. Using the same argument as in the proof of Theorem 1.1, for any maximal ideal  $M$  of  $R[x; \delta]$ , if  $M_0 = M \cap R$ , then  $R/M_0$  is a field and  $\delta(R) \subseteq M_0$ . Let  $I$  be a maximal ideal of  $R$  such that  $R/I$  is a field and  $\delta(R) \subseteq I$ . Then the map

$$\Phi_I: R[x; \delta] \rightarrow (R/I)[x], \quad a_0 + a_1x + \cdots + a_nx^n \mapsto \overline{a_0} + \overline{a_1}x + \cdots + \overline{a_n}x^n$$

is a (well-defined) surjective ring morphism. Moreover,  $\ker \Phi_I = I[x; \delta]$ . Hence,

$$R[x; \delta]/I[x; \delta] \cong (R/I)[x].$$

In particular,  $R[x; \delta]/I[x; \delta]$  is semiprimitive by Lemma 3.1. These facts will be used several times in this section.

**Proposition 4.2.** *Let  $R[x; \delta]$  be a quasi-duo differential polynomial ring, and let  $I$  and  $\Phi_I$  be as in Remark 4.1. Then for any maximal ideal  $N$  of  $S$  containing  $I$ , we have*

$$\Phi_I^{-1}(\Phi_I(N)) = N.$$

*Proof.* Clearly,  $N \subseteq \Phi_I^{-1}(\Phi_I(N))$ . Because of the maximality of  $N$ , we have either  $N = \Phi_I^{-1}(\Phi_I(N))$  or  $\Phi_I^{-1}(\Phi_I(N)) = S$ . If  $\Phi_I^{-1}(\Phi_I(N)) = S$ , then there exists  $a_0 + a_1x + \dots + a_nx^n \in N$  such that

$$\bar{1} = \bar{a}_0 + \bar{a}_1x + \dots + \bar{a}_nx^n.$$

Hence,  $1 - a_0, a_1, \dots, a_n \in I$ . For any  $i \geq 1$  we have  $a_i \in I$  and hence  $a_ix^i \in N$ . This implies that  $a_0 \in N$ . From the fact that  $1 - a_0 \in N$ , we conclude that  $1 \in N$ . This is a contradiction. Hence,  $N = \Phi_I^{-1}(\Phi_I(N))$ .  $\square$

Given a differential polynomial ring  $R[x; \delta]$ , denote by  $\mathcal{M}(R)$  the set of maximal ideals  $I$  of  $R$  such that  $R/I$  is a field and  $\delta(R) \subseteq I$ . Recall that a *monic polynomial* in  $R[x; \delta]$  is an element whose highest degree coefficient is equal to 1. For any  $I \in \mathcal{M}(R)$ , the set of all irreducible monic polynomials in  $(R/I)[x]$  is denoted by  $\mathcal{P}(R/I)$ .

**Theorem 4.3.** *Let  $S = R[x; \delta]$  be a quasi-duo differential polynomial ring. Consider the following two maps:*

- (i) *Associate with any pair  $A = (I, x^n + \overline{a_{n-1}}x^{n-1} + \dots + \overline{a_0}) \in (\mathcal{M}(R), \mathcal{P}(R/I))$  the maximal ideal  $M(A) = I[x, \delta] + \langle x^n + a_{n-1}x^{n-1} + \dots + a_0 \rangle_S$  of  $S$ ;*
- (ii) *Associate with any maximal ideal  $M$  of  $S$  the pair*

$$A(M) = (M_0, p(x)) \in (\mathcal{M}(R), \mathcal{P}(R/M_0))$$

*where  $M_0 = M \cap R$  and  $p(x) \in (R/M_0)[x]$  is such that  $\langle p(x) \rangle_{(R/M_0)[x]} = \Phi_{M_0}(M)$ .*

*Then, these maps yield two mutually inverse bijections between the set of all maximal ideals of  $S$  and the set*

$$\{(I, p(x)) \mid I \in \mathcal{M}(R) \text{ and } p(x) \in \mathcal{P}(R/I)\}.$$

*Proof.* We must first show that the maps from (i) and (ii) are well-defined, that is  $M(A)$  is a maximal ideal of  $S$  for any pair  $A = (I, p(x))$ , and  $A(M)$  is an element of the set  $\{(I, p(x)) \mid I \in \mathcal{M}(R) \text{ and } p(x) \in \mathcal{P}(R/I)\}$  for any maximal ideal  $M$  of  $S$ .

Let  $A$  be a pair  $(I, p(x))$  where  $K = R/I$  is a field,  $p(x) = x^n + \overline{a_{n-1}}x^{n-1} + \dots + \overline{a_0}$  is an irreducible polynomial in  $K[x]$ . We have  $L = \langle p(x) \rangle_{K[x]}$ , i.e. the ideal of  $K[x]$  generated by  $p(x)$ . Since  $p(x)$  is irreducible,  $L$  is maximal in  $K[x]$ . Notice that  $\Phi_I(x^n + a_{n-1}x^{n-1} + \dots + a_0) = p(x)$ . Hence, if  $\Phi_I(f) \in L$ , then  $f \in \langle x^n + a_{n-1}x^{n-1} + \dots + a_0 \rangle_S + I[x; \delta]$  (using that  $R[x; \delta]/I[x; \delta] \cong K[x]$  via  $\Phi_I$  in Remark 4.1). This implies that  $\Phi_I^{-1}(L) = M(A)$ . Again, since  $R[x; \delta]/I[x; \delta] \cong K[x]$  via  $\Phi_I$ , the ideal  $M(A)$  is a maximal ideal of  $R[x; \delta]/I[x; \delta]$ . Notice that  $I[x; \delta] \subseteq M(A)$ , and hence  $M(A)$  is a maximal ideal of  $S$ . Thus, the map from (i) is well-defined.

Now assume that  $M$  is a maximal ideal of  $S$ . Put  $M_0 = M \cap R$ . By Remark 4.1,  $K = R/M_0$  is a field and  $\delta(R) \subseteq M_0$ . Hence,  $M_0 \in \mathcal{M}(R)$ . Since  $M$  is maximal in  $S$ , the

ideal  $\Phi_{M_0}(M)$  is also maximal in  $K[x]$ , by Remark 4.1. Therefore, there exists a unique irreducible monic polynomial  $p(x) \in K[x]$  such that

$$\Phi_{M_0}(M) = \langle p(x) \rangle_{(R/M_0)[x]}.$$

Hence, the map from (ii) is well-defined.

Now we will show that  $M(A(M)) = M$  holds for any maximal ideal  $M$  of  $S$ , and that  $A(M(A)) = A$  holds for any pair  $A \in \{(I, p(x)) \mid I \in \mathcal{M}(R) \text{ and } p(x) \in \mathcal{P}(R/I)\}$ . Let  $M$  be a maximal ideal of  $S$ . Assume that  $A(M) = (M_0, p(x))$  where  $M_0 = M \cap R$  and  $p(x) = x^n + \overline{a_{n-1}}x^{n-1} + \dots + \overline{a_0} \in \mathcal{P}(R/M_0)$ . Notice that  $M_0 \subseteq M$ , so that  $M_0[x; \delta] \subseteq M$ . Therefore,  $x^n + \overline{a_{n-1}}x^{n-1} + \dots + \overline{a_0} \in M$ , which yields that  $M_0[x; \delta] + \langle x^n + \overline{a_{n-1}}x^{n-1} + \dots + \overline{a_0} \rangle_S \subseteq M$ . Equivalently,  $M(A(M)) \subseteq M$ . By the maximality of  $M(A(M))$  in  $S$  we get  $M(A(M)) = M$ .

Let  $A = (I, p(x))$  where  $I$  is an ideal in  $\mathcal{M}(R)$  and  $p(x) = x^n + \overline{a_{n-1}}x^{n-1} + \dots + \overline{a_0} \in \mathcal{P}(R/I)$ . Then,

$$M(A) = I[x; \delta] + \langle x^n + \overline{a_{n-1}}x^{n-1} + \dots + \overline{a_0} \rangle_S.$$

It is clear that  $I = M(A) \cap R$  and that  $\Phi_I(M(A)) = \langle p(x) \rangle_{K[x]}$ . Thus,  $A = A(M(A))$ .  $\square$

We will now use the preceding theorem to describe the Jacobson radical of  $R[x; \delta]$  and obtain a result which resembles [1, Theorem 1.2].

**Corollary 4.4.** *Let  $S = R[x; \delta]$  be a quasi-duo differential polynomial ring. Put  $K = \bigcap_{I \in \mathcal{M}(R)} I$ .*

*Then the Jacobson radical of  $S$  is  $J(S) = K[x; \delta] = (J(S) \cap R)[x; \delta]$ .*

*Proof.* By Theorem 4.3,  $J(S)$  is the intersection of all ideals of the form

$$I[x; \delta] + \langle x^n + \overline{a_{n-1}}x^{n-1} + \dots + \overline{a_0} \rangle_S$$

where  $I$  ranges over  $\mathcal{M}(R)$  and the polynomial  $x^n + \overline{a_{n-1}}x^{n-1} + \dots + \overline{a_0}$  ranges over  $\mathcal{P}(R/I)$ . That is,

$$J(S) = \bigcap_{I \in \mathcal{M}(R)} \bigcap_{x^n + \overline{a_{n-1}}x^{n-1} + \dots + \overline{a_0} \in \mathcal{P}(R/I)} (I[x; \delta] + \langle x^n + \overline{a_{n-1}}x^{n-1} + \dots + \overline{a_0} \rangle_S).$$

For any  $I \in \mathcal{M}(R)$ , we define

$$L_I = \bigcap_{x^n + \overline{a_{n-1}}x^{n-1} + \dots + \overline{a_0} \in \mathcal{P}(R/I)} (I[x; \delta] + \langle x^n + \overline{a_{n-1}}x^{n-1} + \dots + \overline{a_0} \rangle_S).$$

According to the maps in Theorem 4.3, when  $x^n + \overline{a_{n-1}}x^{n-1} + \dots + \overline{a_0}$  ranges over  $\mathcal{P}(R/I)$ , then  $I[x; \delta] + \langle x^n + \overline{a_{n-1}}x^{n-1} + \dots + \overline{a_0} \rangle_S$  ranges over the set of all maximal ideals of  $R[x; \delta]/I[x; \delta]$ . Hence,  $L_I$  is the Jacobson radical of  $R[x; \delta]/I[x; \delta]$ . Thus,  $L_I = I[x; \delta]$  using that  $R[x; \delta]/I[x; \delta]$  is semiprimitive. Therefore,  $J(S) = K[x; \delta] = (J(S) \cap R)[x; \delta]$ .  $\square$

## 5. DIFFERENTIAL POLYNOMIAL RINGS IN SEVERAL INDETERMINATES

In this section we shall show that differential polynomial rings in several indeterminates can never be quasi-duo (see Theorem 5.3).

Let us begin by recalling the definition of a differential polynomial ring in a set of indeterminates. Let  $I$  be a non-empty (possibly infinite) countable set, let  $D = \{\delta_i \mid i \in I\}$  be a family of derivations on  $R$  (by “a family” we mean that all  $\delta_i$ ’s need not be distinct), and let  $X = \{x_i \mid i \in I\}$  be a set of distinct non-commuting indeterminates. Given  $R$ ,  $D$  and  $X$ , we can define the ring  $R[X; D]$  which is the set of all polynomials in the indeterminates  $x_i \in X$  with coefficients from  $R$ . The addition in  $R[X; D]$  is the natural one and the multiplication is generated by the commutation rule  $x_i a = ax_i + \delta_i(a)$ , for  $i \in I$ . The ring  $R[X, D]$  is called a *differential polynomial ring in several indeterminates*. Readers are referred to [2, 12] for more details on this class of rings. In particular, every element  $f \in R[X; D]$  can be written in the form

$$f = a_1 t_1 + a_2 t_2 + \cdots + a_n t_n,$$

where  $a_1, a_2, \dots, a_n \in R \setminus \{0\}$  and  $t_1, t_2, \dots, t_n$  are distinct monomials in  $X$ , i.e. finite words in the alphabet  $X$ . In this case, the support of  $f$  is defined as  $\text{supp}(f) = \{t_1, t_2, \dots, t_n\}$ . If  $\delta_i = 0$ , for all  $i \in I$ , then  $R[X]$  is called a *free polynomial ring*.

A subring  $B$  of a ring  $A$  is called a *corner subring* of  $A$  if  $B$  is unital, possibly with  $1_A \neq 1_B$ , and if there exists an additive subgroup  $C$  of  $A$  such that  $A = B \oplus C$  and  $BC, CB \subseteq C$ . The subgroup  $C$  is called a *complement* of  $B$ . We say that  $B$  is a *left corner* of  $A$  if the complement  $C$  satisfies  $BC \subseteq C$  only. The notion of a *right corner* is defined analogously. A classical example of a corner subring is given by  $B = eAe$ , where  $e \in A$  is an idempotent.

The following lemma shows that quasi-duoness of differential polynomial rings in several indeterminates can be inherited by certain subrings.

**Lemma 5.1.** *Let  $R$  be a ring, let  $I$  be a non-empty countable set, let  $D = \{\delta_i \mid i \in I\}$  be a family of derivations on  $R$ , and let  $X = \{x_i \mid i \in I\}$  be a set of non-commuting indeterminates. For any subset  $J \subseteq I$ , put  $X_J = \{x_i \mid i \in J\}$  and  $D_J = \{\delta_i \mid i \in J\}$ . The ring  $S_J = R[X_J; D_J]$  is a right (left) corner of  $S = R[X; D]$ . In particular, if  $S$  is left (right) quasi-duo, then  $S_J$  is left (right) quasi-duo for any subset  $J \subseteq I$ .*

*Proof.* Take  $J \subseteq I$ . Denote by  $X_J^+$  the set of all nontrivial monomials in  $X_J$ , that is  $X_J^+$  is the set of all “words”  $x_{j_1}^{m_1} x_{j_2}^{m_2} \cdots x_{j_t}^{m_t}$  where  $x_{j_i}$  ranges over  $X_J$ , and  $t, m_i > 0$ . Now put

$$C = \{f \in S \mid \text{supp}(f) \cap X_J^+ = \emptyset\}.$$

To demonstrate that  $S_J$  is a right corner of  $S$ , we will show  $C$  is an additive group,  $S_J \oplus C = S$  and  $CS_J \subseteq C$ . It is easy to show the first statement since  $C$  is an additive subgroup of  $S$  generated by all elements of the following form  $ax_{i_1}^{t_1} x_{i_2}^{t_2} \cdots x_{i_m}^{t_m} \in S$  with  $x_{i_1}^{t_1} x_{i_2}^{t_2} \cdots x_{i_m}^{t_m} \notin X_J^+$ . Again, by the definitions of  $S_J$  and  $C$ , one has  $S = S_J + C$  and  $S_J \cap C = \{0\}$ . Therefore,  $S = S_J \oplus C$ . Now we must show that  $CS_J \subseteq C$ . If  $f = x_{i_1}^{t_1} x_{i_2}^{t_2} \cdots x_{i_m}^{t_m} \in X_J^+$  and  $g = x_{j_1}^{q_1} x_{j_2}^{q_2} \cdots x_{j_l}^{q_l} \notin X_J^+$ , then  $gf \in C$ . Notice that  $S_J$  is the set of



all finite sums  $\alpha f$  where  $\alpha \in R$  and  $f \in X_J^+$ , and that  $C$  is the set of all finite sums  $\beta g$  where  $\beta \in R$  and  $g \notin X_J^+$ . Thus,  $CS_J \subseteq C$ . The last conclusion now follows directly from [7, Theorem 1.2].

Analogously, one can show that  $S_J$  is a left corner of  $S$  and that right quasi-duoness of  $S$  implies right quasi-duoness of  $S_J$ .  $\square$

**Lemma 5.2.** *Let  $I$  be a non-empty countable set and let  $S = R[X; D]$  be a left (right) quasi-duo differential polynomial ring in several indeterminates (as above). If  $R$  is a simple ring, then  $R$  is a field,  $\delta = 0$  and  $|I| = 1$ .*

*Proof.* Suppose that  $S$  is left quasi-duo. (The right quasi-duo case can be treated analogously and is therefore omitted.) Take  $i \in I$ . By Lemma 5.1,  $S_i = K[x_i; \delta_i]$  is left quasi-duo. Using Lemma 2.1 we conclude that  $R$  is a field and that  $\delta_i = 0$ .

It remains to show that  $|I| = 1$ . If  $|I| > 1$ , then it is well-known that the free polynomial algebra  $K[X]$  is left primitive (see e.g. [4, Page 36]). In view of [6, Proposition 4.1],  $K[X]$  is a division ring. This is a contradiction. Therefore,  $|I| = 1$ .  $\square$

**Theorem 5.3.** *Let  $I$  be a non-empty countable set and let  $S = R[X; D]$  be a differential polynomial ring in several indeterminates (as above). If  $S$  is left (right) quasi-duo, then  $|I| = 1$ .*

*Proof.* The proof is essentially the same as the proof of Proposition 2.2, and we will therefore omit some details. As in the proof of Proposition 2.2 we need to consider two cases:

**Case 1.** This will lead to that  $R$  is a division ring. Thus, by Lemma 5.2, we get that  $|I| = 1$ .

**Case 2.** Analogously to the proof of Proposition 2.2 we will define a surjective ring morphism

$$\varphi : R[X; D] \rightarrow (R/M_0)[X; \overline{D}], \quad a_0 + a_1 t_1 + \cdots + a_n t_n \mapsto \overline{a_0} + \overline{a_1} t_1 + \cdots + \overline{a_n} t_n$$

and use it to conclude that  $R/M_0[X; \overline{D}]$  is left (right) quasi-duo. By invoking case 1, we conclude that  $|I| = 1$ .  $\square$

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